

SOLUTION OF A PROBLEM OF AYRES

A Dissertation

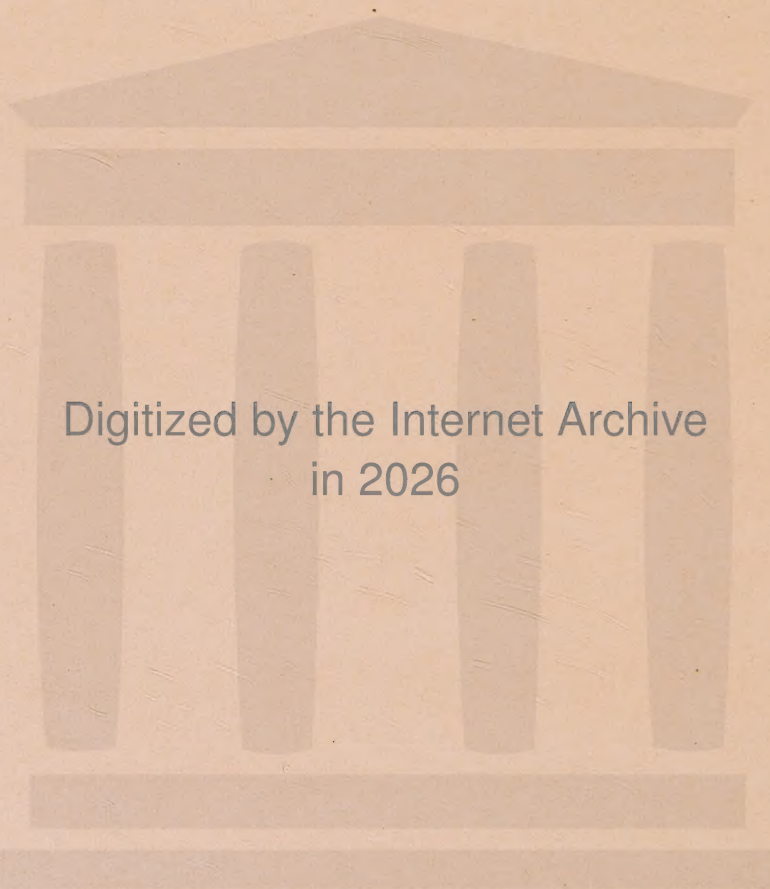
SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY IN
THE UNIVERSITY OF MICHIGAN

BY

LOYAL F. OLLMANN

1941

Reprinted from THE AMERICAN JOURNAL OF MATHEMATICS,
Vol. LXIV, number 1, pages 61 to 71.



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SOLUTION OF A PROBLEM OF AYRES.*

By LOYAL F. OLLMANN

In a paper in 1932, W. L. Ayres¹ suggests the question: If K is any finite subset of a Peano space M which is a subset of the Euclidean plane, and if no two points of K may be separated² by the omission of any two points of M , then does there exist a simple closed curve J such that $M \supset J \supset K$? Ayres answers the question in the affirmative in the same paper for $K = 3$ points, and in later unpublished work he discovered an example which answers the question in the negative for $K \geq 11$ points.

In this paper we prove that the answer to Ayres' question is in the affirmative for $K \leq 5$ points and give examples to show that it is in the negative for $K > 5$ points. We also show that the same results are obtained provided the condition "no two points of K may be separated" is changed to read "no two points of M may be separated." Examples are constructed to show that in certain Peano spaces a finite set K can not be joined by m (where m is any given integer) mutually exclusive simple closed curves.

The writer wishes to express his thanks to Professor W. L. Ayres for his helpful criticisms and generous assistance in the preparation of this paper.

Historical note. The theorem, that if two points of a planar Peano space are not separated by the omission of any single point, then they can be joined by a simple closed curve, was proved by G. T. Whyburn³ and W. L. Ayres.⁴ K. Menger⁵ proved the theorem for regular curves while Ayres⁶ extended the

* Received January 27, 1941. Presented to the American Mathematical Society, April 15, 1939. This paper is a portion of a dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in the University of Michigan.

¹ "On joining finite subsets of a Peano space by arcs and simple closed curves," *Fundamenta Mathematicae*, vol. 19 (1932), pp. 79-91.

² A point set H of a connected set M is said to *separate* the points x and y in M if $M - H$ is the sum of two mutually separated sets containing x and y respectively.

³ "Some properties of continuous curves," *Bulletin of the American Mathematical Society*, vol. 33 (1927), pp. 305-308.

⁴ This result was announced in an abstract "On the separation of points of a continuous curve by arcs and simple closed curves," *Bulletin of the American Mathematical Society*, vol. 33 (1927), p. 266.

⁵ "Zur allgemeinen Kurventheorie," *Fundamenta Mathematicae*, vol. 10 (1927), pp. 96-115.

⁶ "Concerning continuous curves in metric space," *American Journal of Mathematics*, vol. 51 (1929), pp. 577-594.

result to general Peano spaces. Conditions under which three points lie on a simple closed curve have been given by Ayres.⁷

The theorem, that if two closed subsets A and B of a Peano space are not separated by any n points, then they can be joined by $n + 1$ independent arcs,⁸ was proved by N. E. Rutt⁹ for the case where M is a subset of the plane and A and B are single points. Menger¹⁰ proved the theorem for regular curves. It was extended to compact Peano spaces by G. Nöbeling¹¹ and to locally compact Peano spaces by Leo Zippin.¹² Ayres¹³ proved that if K is a set consisting of n points ($n > 1$) of a Peano space P such that no two points of K may be separated by the omission of $n - 1$ points of P , then there exists a simple closed curve of P containing K .

Notation and preliminary lemmas. Throughout this paper we assume that K is a finite subset of a Peano space M , a subset of the Euclidean plane E_2 . We say that a set K is *3-connected in M* if there do not exist two points x and y of M such that $M - (x + y) = M_1 + M_2$, where M_1 and M_2 are mutually separated and $M_1K \neq 0$, $M_2K \neq 0$. This term seems appropriate since between any two points of K there are three independent arcs in M . It is evident that if K is 3-connected in M , then every point of K is of Menger-Urysohn order ≥ 3 . We also say that if for every two points x and y , the set $M - (x + y)$ is connected, then M is *3-connected*, in which case every point of M is of order ≥ 3 . The letter J will be used to denote a simple closed curve contained in M . In general, capital letters will denote sets of points, while lower case letters denote individual points or numbers.

LEMMA 1. *If K is 3-connected in M , and $p_1 \in K$ is interior to the simple closed curve $J \subset M$, and $p_2 \in K$ is exterior to J , then there exist in M three arcs ρ_1 , ρ_2 , and ρ_3 which join p_1 to J such that $\rho_i \cdot \rho_j$ is the point p_1 ($i \neq j$).*

The lemma follows as a direct result of the theorem due to Rutt.¹⁴ For

⁷ "Continuous curves which are cyclicly connected," *Bulletin de l'Académie Polonaise des Sciences et des Lettres*, (1928), pp. 127-142.

⁸ Arcs which are mutually exclusive except possibly for their common end-points are called *independent arcs*.

⁹ "Concerning the cut points of a continuous curve when the arc curve, AB , contains exactly N independent arcs," *American Journal of Mathematics*, vol. 51, No. 2 (1929), pp. 217-246.

¹⁰ *Loc. cit.*

¹¹ "Eine Verschärfung des n -Beinsatzes," *Fundamenta Mathematicae*, vol. 18 (1932), pp. 23-38.

¹² "Independent arcs of a continuous curve," *Annals of Mathematics*, vol. 34, No. 1 (1933), pp. 95-113.

¹³ "On joining finite subsets of a Peano space by arcs and simple closed curves," *loc. cit.*⁹

¹⁴ *Loc. cit.*⁹ Theorem IV.

let α_1 , α_2 , and α_3 be three independent arcs which join p_1 and p_2 . By the Jordan curve theorem¹⁵ each of these arcs has at least one point in common with J . Let a_i ($i = 1, 2, 3$) be the first point in the order from p_1 to p_2 which α_i has in common with J . Since the α_i are independent, the points a_i are distinct. Let ρ_i be the subarc $p_1 a_i$ of α_i , then $\rho_i \cdot \rho_j = p_1$, and the lemma is proved.

LEMMA 2. *If K is 3-connected in M , and $p_1 \in K$ is interior to the simple closed curve $J \subset M$, and $p_2, p_3 \in K$ lie on J , then either there exist (1) three arcs ρ_1, ρ_2 , and ρ_3 which join p_1 to J such that $\rho_i \cdot \rho_j = p_1$ ($i, j = 1, 2, 3; i \neq j$), or (2) two arcs ρ_1 and ρ_2 such that $\rho_1 \cdot J = p_2, \rho_2 \cdot J = p_3$, and $\rho_1 \cdot \rho_2 = p_1$.*

Proof. Let α_i ($i = 1, 2, 3$) denote three independent arcs from p_1 to p_2 , and let a_i be the first point of J on α_i in the order p_1 to p_2 . If the a_i are distinct points, the lemma is true for this case. If $a_i = a_j$ ($i \neq j$), then it is clear from the definition of independent arcs that $a_i = a_j = p_2$. Let β_i ($i = 1, 2, 3$) denote three independent arcs joining p_1 to p_3 , with b_i the first point of J on β_i in the order p_1 to p_3 . If the b_i are distinct points then the result is obvious. Therefore we may assume that $a_1 = a_2 = p_2$ and $b_1 = b_2 = p_3$. In the order p_3 to p_1 let b_1' be the first point of $\alpha_1 + \alpha_2$ on β_1 , and suppose that b_1' belongs to α_1 . Let $\rho_1 = \alpha_2$ and $\rho_2 =$ subarc $p_1 b_1'$ of $\alpha_1 +$ subarc $b_1' p_3$ of β_1 , which satisfies conclusion (2) of the lemma.

LEMMA 3. *If K is 3-connected in M , and $p_1 \in K$ is interior to the simple closed curve $J \subset M$, and p_2, p_3 , and $p_4 \in K$ lie on J , then there exist three arcs ρ_1, ρ_2 , and ρ_3 which join p_1 to J such that $\rho_i \cdot \rho_j = p_1$ ($i \neq j$).*

Proof. Using the theorem of Rutt, we let α_i ($i = 1, 2, 3$) denote the three independent arcs from p_1 to p_2 . Let a_i ($i = 1, 2, 3$) represent the first points of α_i on J in the order p_1 to p_2 . If the a_i are distinct points, then the conclusion of the lemma follows. If two of the points a_i coincide, then the independence of the arcs α_i requires the point of coincidence to be p_2 . We may thus suppose $a_1 = a_2 = p_2$. It is not important in this discussion whether a_3 and p_2 coincide or not. Because of Lemma 2 there are either three independent arcs from p_3 and p_4 to p_1 (in which case the conclusion of this lemma is immediate) or there exist two independent arcs from p_3 and p_4 respectively to p_1 . We assume that β joins p_3 to p_1 and γ joins p_4 to p_1 . In the order p_3 to p_1 on β , let b represent the first point of β on $\Sigma \alpha_i$, and let c be the first point of γ on $\Sigma \alpha_i$ in the order p_4 to p_1 . If b and c lie on different arcs, say α_1 and α_3 , respectively, then $p_1 b$ is a subset of α_1 , $b p_3 \subset \beta$, $p_1 c \subset \alpha_3$, and $c p_4 \subset \gamma$. Thus $p_1 b + b p_3$, $p_1 c + c p_4$, and α_2 are the required

¹⁵ C. Jordan, *Cours d'Analyse*, Paris (1893), p. 90.

arcs, and they would also satisfy the lemma if $b = p_1$ or $c = p_1$ or both $b, c = p_1$.

We now assume that b and c lie on the same arc and for our purpose we assume this arc to be α_3 . We further assume the order on α_3 to be p_1cba_3 . In the order b to p_1 on the subarc bp_1 of β , let b_1 represent the first point on β of the point-set $\alpha_1 + \alpha_2$. In like manner let c_1 represent the first point of $\alpha_1 + \alpha_2$ on the arc cp_1 of γ in the order c to p_1 . Either b_1 or c_1 or both may be the point p_1 but in these cases the arcs are quite evident. For example, if c_1 lies on α_2 and $b_1 = p_1$, then α_1 , subarc p_1c_1 of $\alpha_2 +$ subarc c_1p_4 of γ , and β are the three desired arcs. The two cases remaining to be discussed are those in which b_1 and c_1 lie on different arcs or on the same arc.

We suppose that b_1 lies on α_2 and c_1 lies on α_1 , both points being distinct from p_1 . In the order p_1 to a_3 on α_3 , let x denote the first point of α_3 on the sum of the subarcs cc_1 of γ and bb_1 of β . If x belongs to β then subarc p_1c_1 of $\alpha_1 +$ subarc c_1p_4 of γ , α_2 , and subarc p_1x of $\alpha_3 +$ subarc xp_3 of β , are three independent arcs from p_1 to J . If x is on γ then subarc p_1x of $\alpha_3 +$ subarc xp_4 of γ , subarc p_1b_1 of $\alpha_2 +$ subarc b_1p_3 of β , and α_1 are the three desired arcs.

We now suppose that b_1 and c_1 both lie on α_2 (the case for α_1 is similar). Assume the order on α_2 to be $p_2b_1c_1p_1$. In the order p_1 to a_3 on α_3 , let x denote the first point on p_1c of subarc c_1c of γ . The arc p_1x does not have a point in common with the subarc b_1p_3 as may be seen by application of the Jordan curve theorem to the various cases which may arise. The three arcs satisfying this case are: α_1 , subarc p_1x of $\alpha_3 +$ subarc xp_4 of γ , and subarc p_1b_1 of $\alpha_2 +$ subarc b_1p_3 of β . If the order on α_2 is $p_2c_1b_1p_1$ let y be the first point in the order p_1 to a_3 on α_3 of subarc b_1b of β . As above, p_1y does not have a point in common with c_1c of γ . In this case the three arcs α_1 , subarc p_1y of $\alpha_3 +$ subarc yp_3 of β , and subarc p_1c_1 of $\alpha_2 +$ subarc c_1p_4 of γ are the required arcs. This completes the proof of the lemma.

LEMMA 4. *If K is 3-connected in M , then for any three points $p_1, p_2, p_3 \in K$, there exist three independent arcs α_1, α_2 , and α_3 with end-points p_1 and p_2 such that p_3 belongs to one of the arcs α_i .*

Proof. Let α_1, α_2 , and α_3 be any three independent arcs which join p_1 and p_2 . If p_3 belongs to one of the arcs α_i , then the lemma is proved. Suppose then that p_3 is interior to one of the domains complementary to $\alpha_1 + \alpha_2 + \alpha_3$ whose boundary we assume to be $J' = \alpha_1 + \alpha_2$, a simple closed curve. By Lemma 2, there are either (1) three independent arcs β_1, β_2 , and β_3 that join p_3 to J' such that the end-points b_i are distinct, or (2) two independent arcs δ_1 and δ_2 such that $\delta_1 \cdot J' = p_1$, $\delta_2 \cdot J' = p_2$, and $\delta_1 \cdot \delta_2 = p_3$. In (1) at least two of the points b_i belong to either α_1 or α_2 . Suppose b_1 and

b_2 belong to α_1 in the order $p_1 b_1 b_2 p_2$. The point set $p_1 b_1 + \beta_1 + \beta_2 + b_2 p_2$ is an arc which contains p_3 , and with α_2 and α_3 forms a set of independent arcs joining p_1 and p_2 . In (2), we merely let $\delta_1 + \delta_2$ replace α_1 in the independent set α_i , which proves the lemma.

THEOREM 1. *Let K be any set of ≤ 5 points which is 3-connected in M , then there exists a simple closed curve J such that $M \supset J \supset K$.*

Proof. Let K be the five points $(p_1, p_2, \dots, p_5)$. By Lemma 4 there exist three independent arcs, α_1, α_2 , and α_3 , in M which join p_1 and p_2 and contain p_3 . We let α_1 be the arc which contains p_3 . Let D_1, D_2 , and D_3 be the complementary domains of $\alpha_1 + \alpha_2 + \alpha_3$ such that the frontier of D_i does not contain α_i . Because of the symmetry of the points p_i and the arcs α_i , it is necessary to discuss only the following cases:

Case I. The points p_4 and p_5 lie on the arcs α_i .

I_a . One of the arcs α_2 or α_3 contains neither the point p_4 nor the point p_5 .

I_b . Each arc α_i contains one of the points.

Case II. One arc contains p_4 , while p_5 is an interior point of one of the complementary domains D_i .

II_a . p_4 is on α_1 , p_5 is in D_1 .

II_b . p_4 is on α_1 , p_5 is in D_2 or D_3 .

II_c . p_4 is on α_2 , p_5 is in D_3 .

II_d . p_4 is on α_2 , p_5 is in D_1 or D_2 .

Case III. Both points are interior points of the complementary domains D_i .

III_a . Both p_4 and p_5 are in D_2 or D_3 .

III_b . Both p_4 and p_5 are in D_1 .

III_c . p_4 is in D_3 , p_5 is in D_2 .

III_d . p_4 is in D_2 or D_3 , p_5 is in D_1 .

Case I_a . Let α_3 be that arc which does not contain the points p_4 or p_5 . Then $\alpha_1 + \alpha_2$ is a simple closed curve containing K .

Case I_b . Suppose α_2 contains p_4 and α_3 contains p_5 . Let α'_1 and α''_1 denote the subarcs $p_3 p_1$ and $p_3 p_2$ of α_1 . In the same manner, let p_4 divide α_2 into subarcs α'_2 and α''_2 , and p_5 divide α_3 into α'_3 and α''_3 . Due to symmetry we also suppose that D_2 is the unbounded domain. Since $p_4 \in K$ is interior to the simple closed curve $\alpha_1 + \alpha_3$, there exist (by Lemma 3) three independent arcs β_1, β_2 , and β_3 which join p_4 to distinct points b_1, b_2 , and b_3 , respectively,

of $\alpha_1 + \alpha_3$. If two of the b_i belong to the same subarc α'_1 , α''_1 , α'_3 , or α''_3 , then it is evident that K is contained in a simple closed curve.

Suppose, however, that b_1 is on α'_1 , b_2 is on α''_1 , and b_3 is on α'_3 . Every other distribution will be similar to this case. In the order from p_1 to p_4 , let x' be the first point on α'_2 which is also a point of $\Sigma\beta_i$. Denote the subarc $x'p_4$ of β_i by β'_i . Then the following cases represent the possible distributions of x' on the β_i with a simple closed curve containing K , thus satisfying the theorem:

- 1) x' on β_1 : $p_1x' + \beta'_1 + \beta_3 + b_3p_5 + \alpha''_3 + \alpha_1 = J$
- 2) x' on β_2 : $p_1x' + \beta'_2 + \beta_1 + b_1p_3 + \alpha''_1 + \alpha_3 = J$
- 3) x' on β_3 : $p_1x' + \beta'_3 + \beta_1 + b_1p_3 + \alpha''_1 + \alpha_3 = J$

This completes the proof for case I_b .

Case II_a. By Lemma 1, there exist three independent arcs β_1 , β_2 , and β_3 which join p_5 to the distinct points b_1 , b_2 , and b_3 of $\alpha_2 + \alpha_3$. Two of the points b_i must lie on α_2 or on α_3 . Suppose b_1 and b_2 lie on α_2 in the order $p_1b_1b_2p_2$. Then let J be $\alpha_1 + p_1b_1 + \beta_1 + \beta_2 + b_2p_2$. The case for b_1 and b_2 on α_3 is similar.

Case II_b. Suppose p_5 is in D_3 , the other case being similar to it. We also assume the order $p_1p_3p_4p_2$ on α_1 . Let β_1 , β_2 , and β_3 be three independent arcs (Lemma 3) with end-points p_5 and b_i ($i = 1, 2, 3$), where the b_i are distinct points of the simple closed curve $\alpha_1 + \alpha_2$. If two of the b_i , say b_1 and b_2 , lie on α_2 in the order $p_1b_1b_2p_2$, then $p_1b_1 + \beta_1 + \beta_2 + b_2p_2 + \alpha_1 = J$ is a simple closed curve containing K . Let α'_1 , α''_1 , and α'''_1 denote the subarcs p_1p_3 , p_3p_4 , and p_4p_2 of α_1 . If b_1 and b_2 lie on α'_1 then $p_1b_1 + \beta_1 + \beta_2 + b_2p_3 + \alpha''_1 + \alpha'''_1 + \alpha_2$ is the desired simple closed curve J ; and similarly for two of the b_i on the arcs α''_1 or α'''_1 . We now assume that each of the arcs α'_1 , α''_1 , α'''_1 , and α_2 contains at most one of the points b_i . If one end-point, say b_3 , is on the arc α_2 , then either b_1 or b_2 belongs to α'_1 or α''_1 for otherwise both would belong to α'''_1 . Suppose that b_1 is on α'_1 . Then K would be contained in the simple closed curve $p_1b_3 + \beta_3 + \beta_1 + b_1p_3 + \alpha''_1 + \alpha'''_1 + \alpha_3$. For b_2 on α'''_1 , J is obtained similarly.

The case remaining is that of b_1 on α'_1 , b_2 on α''_1 , and b_3 on α'''_1 . For the discussion of this case, assume that D_1 is the unbounded complementary domain of $\alpha_1 + \alpha_2 + \alpha_3$ for, if necessary, we can perform an inversion. By Lemma 3, let γ_1 , γ_2 , and γ_3 be three independent arcs with end-points p_4 and c_i ($i = 1, 2, 3$), respectively, such that the c_i are distinct points on the simple closed curve $J' = \alpha_3 + \alpha'_1 + p_3b_2 + \beta_2 + \beta_3 + b_3p_2$ of which p_4 is an interior

point. If two of the points c_i lie on the same subarc α_3 , α'_1 , p_3p_5 , or p_5p_2 of J' , then the simple closed curve which contains K is evident. If one of the points c_i , say c_1 , is on $\langle \alpha_3 \rangle$ ¹⁰, then let x be the first point (possibly p_4), in the order from c_1 to p_4 , which γ_1 has in common with the subarc b_2b_3 of α_1 . Suppose x belongs to b_2p_4 , then $\alpha'_1 + p_3b_2 + \beta_2 + \beta_3 + b_3p_4 + p_4x + xc_1 + c_1p_2 + \alpha_2$ is a simple closed curve containing K ; and similarly for x belonging to p_4b_3 . Thus when one of the c_i is on the arc $\langle \alpha_3 \rangle$ there is a simple closed curve J which has properties required for the conclusion of the theorem.

We assume next that each of the subarcs p_1p_3 , p_3p_5 , and p_5p_2 of J' contains one of the c_i . Suppose c_1 is on p_1p_3 , c_2 is on p_3p_5 and c_3 is on p_5p_2 . For convenience in notation, and because the arcs b_2p_4 and b_3p_4 of α_1 do not enter our discussion, we let $c_2 = b_2$ and $c_3 = b_3$. We assume also that $c_1 = b_1$, which assumption merely narrows the availability of the points b_1 and c_1 as interior points of distinct arcs in a simple closed curve. Denote the simple closed curve $\gamma_1 + \beta_1 + \beta_2 + \gamma_2$ by J'' . If any one of the conditions $c_1 = p_3$, $c_2 = p_3$, or $c_2 = p_5$ are true, the simple closed curve containing K is evident. Thus we assume that the point p_3 is an interior point of J'' , and by use of Lemma 1, there are three independent arcs δ_1 , δ_2 , and δ_3 which join p_3 to three distinct points d_1 , d_2 , and d_3 respectively, of J'' . We consider the possible distributions of the points d_i on the arcs $\gamma_1 = p_4c_1$, $\beta_1 = c_1p_5$, and $\gamma_2 + \beta_2 = p_4p_5$, whose sum is J'' . If two of the d_i are on one of the arcs γ_1 , β_1 , or $\gamma_2 + \beta_2$, then the simple closed curve is again evident; e. g., when d_1 and d_2 are on γ_1 then $p_2c_3 + \beta_3 + \beta_2 + \gamma_2 + p_4d_2 + \delta_2 + \delta_1 + d_1c_1 + c_1p_1 + \alpha_2$ contains K . If no one of the arcs of J'' contains two of the d_i , then each arc contains exactly one. Assume that d_1 is on γ_1 , d_2 is on β_1 , and d_3 is on $\gamma_2 + \beta_2$. Then $p_2c_3 + \beta_3 + \beta_2 + \gamma_2 + p_4d_1 + \delta_1 + \delta_2 + d_2c_1 + c_1p_1 + \alpha_2$ is a simple closed curve containing K . This completes the proof for case II_b .

Case II_c . Let β_1 , β_2 , and β_3 be three independent arcs joining p_5 to b_1 , b_2 , and b_3 respectively on the boundary of D_3 . Let the subarcs p_1p_3 and p_3p_2 of α_1 be α'_1 and α''_1 . Similarly let p_4 divide α_2 into the subarcs α'_2 and α''_2 . The simple closed curve containing K is evident if two of the points b_i fall on the same subarc of α_1 or α_2 . Suppose b_1 is on α'_1 , b_2 is on $\alpha''_1 - p_2$, and b_3 is on α'_2 , any other combination being similar. Then $p_1p_3 + p_3b_2 + \beta_2 + \beta_3 + b_3p_4 + \alpha''_2 + \alpha_3$ is a simple closed curve containing K .

If $b_2 = p_2$, and $b_3 = p_4$, then the simple closed curve $\alpha'_2 + \beta_3 + \beta_1 + b_1p_3 + \alpha''_1 + \alpha_3$ will contain K . We now suppose that $b_2 = p_2$ and $b_3 \neq p_4$. Then p_4 is an interior point of the simple closed curve $p_1b_3 + \beta_3 + \beta_2 + \alpha_3$

¹⁰ If α is an arc with end-points x and y , then by the symbol $\langle \alpha \rangle$ we shall mean $\alpha - (x + y)$.

which we denote by J''' . Then let γ_1, γ_2 , and γ_3 denote the three arcs (Lemma 3) which join p_4 to distinct points c_1, c_2 , and c_3 , respectively, of J''' . If two of the points c_i fall on one of the arcs $p_1b_3 + \beta_3, \beta_2$, and α_3 , their end points included, then the simple closed curve is evident. We thus assume that c_1 lies on $\langle p_1b_3 + \beta_3 \rangle$, c_2 on $\langle \beta_2 \rangle$, and c_3 on $\langle \alpha_3 \rangle$. Then $p_1b_3 + \beta_3 + \beta_1 + b_1p_3 + \alpha''_1 + p_2c_3 + \gamma_2 + \gamma_3 + c_3p_1$ is the desired simple closed curve.

Case II_a. Here again we need discuss only the case where p_5 is in D_1 . Consider the end-points b_i ($i = 1, 2, 3$) of the independent arcs β_i from p_5 to the simple closed curve $\alpha_2 + \alpha_3$. If two of the points, say b_1 and b_2 , belong to α_3 , assuming the order $p_1b_1b_2p_2$ on α_3 , then let $\bar{\alpha}_3$ be the arc $p_1b_1 + \beta_1 + \beta_2 + b_2p_2$. Now α_1, α_2 , and $\bar{\alpha}_3$ each contain a point of K as an interior point and the conditions of Case I_b are satisfied. If two of the b_i fall on α'_2 or α''_2 of α_2 , then let $\bar{\alpha}_2$ be defined similarly to $\bar{\alpha}_3$. The arc $\bar{\alpha}_2$ contains p_4 and p_5 as interior points and the simple closed curve $\alpha_2 + \alpha_1$ contains K . If no two points b_i fall on one of the arcs α_3, α'_2 , or α''_2 , then each arc contains exactly one of the b_i . Suppose b_1 is on α_3 , and b_2 is on α'_2 , then let $J = p_1b_1 + \beta_1 + \beta_2 + b_2p_4 + \alpha''_2 + \alpha_1$. This proves the theorem for Case II .

Case III_a. Assume that p_4 and p_5 are in D_3 , the other possibility being symmetrical. Let the three independent arcs β_i join p_4 to b_i on $\alpha_1 + \alpha_2$. If two of the b_i belong to the same subarc of α_1 , then define $\bar{\alpha}_1$ as above to include p_3 and p_4 . Then the situation is covered by II_b unless p_5 belongs to $\bar{\alpha}_1$. In this event let $J = \bar{\alpha}_1 + \alpha_2$. If two of the b_i belong to α_2 , then define $\bar{\alpha}_2$ to include p_4 and apply II_c or II_d unless p_5 belongs to $\bar{\alpha}_2$. In this case let $J = \bar{\alpha}_2 + \alpha_1$. Finally suppose b_1 is on p_1p_3 of α_1 , b_2 is on p_3p_2 of α_1 , and b_3 is on α_2 . If p_5 is on one of the arcs β_i , the simple closed curve which contains K is apparent.

The arcs β_i divide D_3 into three regions which we will denote by $D', D'',$ and D''' . Since p_5 is not on the boundary of one of these regions it must be an interior point of one of them. Suppose that D' is bounded by $J' = \beta_1 + \beta_3 + b_3p_1 + p_1b_1$, D'' by $J'' = \beta_1 + \beta_2 + b_2p_3 + p_3b_1$, and D''' by $J''' = \beta_2 + \beta_3 + b_3p_2 + p_2b_2$. The discussion for p_5 in D''' is the same as that for p_5 in D' . Suppose now that p_5 is in D' . Let γ_1, γ_2 , and γ_3 be three independent arcs joining p_5 to c_1, c_2 , and c_3 on J' . If two of the c_i belonging to the same subarc of J' , then it is evident that there is a simple closed curve containing K . In the contrary case, suppose that c_1 is on b_1p_1 , c_2 is on β_1 , and c_3 is on β_2 . Then $p_1c_1 + \gamma_1 + \gamma_2 + c_2b_1 + b_1p_3 + p_3b_2 + \beta_2 + \beta_3 + b_3p_2 + \alpha_3$ contains the set K . For other combinations of the c_i on the four subarcs of J' , the simple closed curve containing the set K can be read similarly.

For p_5 in D'' , we proceed as follows. If two of the points c_i , which are

again end-points of independent arcs γ_i , belong to the same subarc of J'' , then J is clearly seen. If no two of the c_i belong to the same subarc of J'' , then one point must be on either p_3b_1 or b_2p_3 , while another one must belong to β_1 or to β_2 . Suppose that c_1 is on p_3b_1 , and that c_2 belongs to β_1 . Then the simple closed curve $p_2b_2 + b_2p_3 + p_3c_1 + \gamma_1 + \gamma_2 + c_2p_4 + \beta_3 + b_3p_1 + \alpha_3$ contains K ; and similarly for other combinations of arcs. This concludes the proof for this case.

Case III_b. For p_4 and p_5 belonging to D_1 , we define $\bar{\alpha}_2$ or $\bar{\alpha}_3$ (depending on which contains at least two of the b_i) to include p_4 . The case is then the same as II_c or II_d , unless $\bar{\alpha}_2$ or $\bar{\alpha}_3$ contains p_5 when the simple closed curve containing K is obvious.

Case III_c. We shall prove this case by reducing it to one of the previous cases. Let β_1, β_2 , and β_3 be three independent arcs joining p_4 to $\alpha_1 + \alpha_2$, with end-points b_i . If two of the b_i belong to the same subarc p_1p_3, p_3p_2 , or to α_2 , then either α_1 or α_2 can be redefined to include p_4 . The situation now is the same as II_b or II_d . Suppose then that p_1p_3 contains b_1 , p_3p_2 contains b_2 , and α_2 contains b_3 . Let $\bar{\alpha}_1 = p_1b_1 + \beta_1 + \beta_2 + b_2p_2$; then α_1, α_2 , and α_3 will be an independent set of arcs joining p_1 and p_2 , such that p_4 belongs to $\bar{\alpha}_1$ and $p_3 + p_5$ lie together in one of the complementary domains of $\bar{\alpha}_1 + \alpha_2 + \alpha_3$. This satisfies the hypothesis of III_a . Thus case III_c is proved.

Case III_d. Let β_1, β_2 , and β_3 be independent arcs joining p_5 to the boundary of $D_1 = \alpha_2 + \alpha_3$. Two of the end-points b_i of β_i lie on either α_2 or α_3 . Then as above, define $\bar{\alpha}_2$ or $\bar{\alpha}_3$ to include p_5 and the case reduces to II_c or to II_d . This completes the proof for $K = 5$ points.

(NOTE. The theorem due to Ayres¹⁷ shows that Theorem 1 is true for $K = 2$ or 3 points, and we have shown the existence of J for $K = 5$ points. For the case of $K = 4$ points we proceed exactly as in the case for 5 points but, having fewer points to consider, the analysis is much simpler.)

THEOREM 2. *If M is 3-connected, and K consists of ≤ 5 points, then there exists a simple closed curve J such that $M \supset J \supset K$.*

Proof. If M is 3-connected then any subset of M is 3-connected in M . Hence Theorem 2 is a direct corollary of Theorem 1.

We now show by means of several examples that the conditions imposed on Theorem 1 can not be weakened.

¹⁷ *Loc. cit.*¹

Example 1. We show here that Theorem 1 is not true for $K \geq 6$ points. Let a_1, a_2 , and a_3 be the vertices of an equilateral triangle T . Let T' be the triangle formed by joining the midpoints of the sides of T . Let x be a point exterior to T , and y be a point interior to T' . Draw independent arcs xa_i ($i = 1, 2, 3$) and yb_i ($i = 1, 2, 3$) where the b_i are the midpoints of the sides of T' . It is easily seen that $K = \Sigma a_i + \Sigma b_i$ is 3-connected in M and that there is no simple closed curve in M which contains the point set K . In this example the points of K are all of order 3, while the points of M are all of order ≤ 4 .

Example 2. In this example we wish to show that Theorem 2 cannot be extended to include more than 5 points in K . Let a circle C be inscribed in an equilateral triangle T with vertices x_1, x_2 , and x_3 . Let the points of tangency be p_1, p_2 , and p_3 . Denote the triangular domains of $T - C$ by C_1, C_2 , and C_3 . In the circle C construct three independent arcs α_1, α_2 , and α_3 with end-points p_1 and p_2, p_2 and p_3, p_3 and p_1 respectively. In the domain D whose boundary is $\Sigma \alpha_i$, construct three arcs β_1, β_2 , and β_3 which are distinct except for their end-points p_1 and p_2, p_2 and p_3, p_3 and p_1 . Denote the region between α_i and β_i by D_i ($i = 1, 2, 3$).

Let b_i denote the midpoints of the arcs β_i , and construct distinct arcs $b_i b_{i+1} \pmod{3}$ in the interior of the domain bounded by $\Sigma \beta_i$. Let B denote the domain whose boundary is $\Sigma b_i b_{i+1}$. Then we let M denote the point set $\Sigma \bar{C}_i + \Sigma \bar{D}_i + \bar{B}$ where the vertices x_1, x_2 , and x_3 are now identified. If c_i is a point of C_i and d_i is a point of D_i , then let K be the six points $\Sigma c_i + \Sigma d_i$. It is evident that M is 3-connected and that every point of M is of the order of the continuum, but it is not possible for K to be included in a simple closed curve in the set M . This example also shows that a previously announced theorem¹⁸ by Ayres is not true for $K > 5$ points.

Example 3. The following example¹⁹ shows that if the condition that M be planar is removed, then Theorem 1 is not true even for K consisting of but four points. In the Euclidean 3-dimensional space, let p_i ($i = 1, \dots, 4$) be the points $(1, 0, 0)$, $(0, 0, 1)$, $(0, 0, -1)$, and $(-1, 0, 0)$, and let q_j ($j = 1, 2, 3$) be the points $(0, -1, 0)$, $(0, 0, 0)$, and $(0, 1, 0)$. Let $M = \Sigma p_i q_j$, where $p_i q_j$ denotes the straight line interval from p_i to q_j , and let $K = \Sigma p_i$. It is easy to see that K is 3-connected in M , and cannot be joined by a simple closed curve in M .

¹⁸ The statement of the theorem is the same as that of Theorem 2 except that K is any finite set. This result was announced in an abstract No. 51, p. 772 *Bulletin of the American Mathematical Society*, vol. 35 (1929), but never published as an error was found in the proof.

¹⁹ Ayres, *loc. cit.*¹

Example 4. Suppose we allow K to be only 2-connected in M , by taking M to be the θ -curve consisting of the three independent arcs xa_iy and let $K = \Sigma a_i$. Then K cannot be joined by a simple closed curve in M .

Example 5. We show here that if M is not a Peano space, even though it be arc-wise²⁰ connected, and K is 3-connected in M and consists of only 2 points, then K may not be included in a simple closed curve in M . In the Euclidean plane, let S denote the curve $y = \sin \pi/x$ ($0 < x \leq 1$). We add the limit points and an arc α which joins the points $(1, 0)$ and $(0, 1)$ but otherwise is distinct from S . Let $M = \bar{S} + \alpha$, and let K be the two points $(0, 1)$ and $(0, -1)$. We observe that K is 3-connected in M , and that M is arc-wise connected, but contains no simple closed curve whatever. Therefore K is not contained in a simple closed curve. This example shows further that Rutt's theorem²¹ is not true if M is not a Peano space, even though it be arc-wise connected.

THEOREM 3. *For any given positive number $m \geq 3$, there exists a planar Peano space M containing a set $K = 2m$ points which is 3-connected in M and cannot be included in $m - 2$ mutually exclusive simple closed curves in M .*

The set M which we shall construct is a generalization of Example 1, in which case $m = 3$. Let N be a regular m -gon with vertices $a_1, a_2, \dots, a_m$. Let N' be the regular m -gon formed from N by joining the midpoints of adjacent sides by lines. Denote the midpoints of the sides of N' by $b_1, b_2, \dots, b_m$ and the vertices of N' by $z_1, z_2, \dots, z_m$. Let y be an interior point of N' and x be a point exterior to N . Draw m independent arcs xa_i and m independent arcs $y b_i$. Let $M = \Sigma xa_i + \Sigma y b_i + N + N'$ and let $K = \Sigma a_i + \Sigma b_i$.

We shall now show that K can not be included in $m - 2$ mutually exclusive simple closed curves. From the construction it is evident that every simple closed curve in M which contains s points of K will also contain s points of the set $Z = x + y + \Sigma z_i$. Since there are only $m + 2$ points in Z , then, irrespective of the manner in which the mutually exclusive simple closed curves are constructed, they can contain at most $m + 2$ points of K . There remain then at least $m - 2$ isolated points of K which can be included only in degenerate simple closed curves. Assuming the most favorable case, namely that $m + 2$ points are contained in a simple closed curve (this can be achieved), then $m - 1$ mutually exclusive simple closed curves are necessary to contain K .

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²⁰ A point set M is *arc-wise* connected when every two points of M are the extremities of a simple continuous arc lying wholly in M .

²¹ *Loc. cit.*⁹

